Matrix

Def Matrix is an array of numbers.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Each number a_{ij} is called entry or element of the matrix. The first subscript i is the row index, and the second subscript j is column index.
The matrix contains m rows and n columns. The order of the matrix is $m \times n$.

If $m = n$ it is called a square matrix.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{22} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \leftarrow \text{diagonal.}$$

Ex:

$$\begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}$$

square matrix
order 2×2

b. $\begin{bmatrix} 1 & -2 & 5 \\ -1 & 3 & 4 \end{bmatrix}$ rectangular matrix
order 2×3

(c) $\begin{bmatrix} 4 & 1 & 11 & 3 \end{bmatrix}$

order 1×4

(d) $\begin{bmatrix} 3/9 & 7 \\ 2 & 1 \\ 6 & 0 \end{bmatrix}$ order : 4×2

A matrix with only one column $\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix}$ is called
a column matrix.

A matrix with only one row $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix}$
is called row matrix

Motivation

Matrices should be interpreted as linear mappings and the subject which studies this is called Linear Algebra (usually taken along with or after Calculus).

Unfortunately, we will study matrices only as a tool to solve system of linear equations. This is how it arose in the first place.

Augmented matrix of a system of linear equations

$$\begin{aligned} 3x + 4y &= 1 \\ x - 2y &= 7 \end{aligned} \longrightarrow \left[\begin{array}{cc|c} 3 & 4 & 1 \\ 1 & -2 & 7 \end{array}\right]$$

$$\begin{aligned} x - y + z &= 2 \\ 2x + 2y - 3z &= -3 \\ x + y + z &= 6 \end{aligned} \longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 2 & 2 & -3 & -3 \\ 1 & 1 & 1 & 6 \end{array}\right]$$

$$\begin{aligned} x + y + z &= 0 \\ 3x - z &= 2 \end{aligned} \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 3 & 0 & -1 & 2 \end{array}\right]$$

$$\begin{aligned} 3x - 2y + 4z &= 5 \\ y - 3z &= -2 \\ 7x - z &= 1 \end{aligned} \longrightarrow \left[\begin{array}{ccc|c} 3 & -2 & 4 & 5 \\ 0 & 1 & -3 & -2 \\ 7 & 0 & -1 & 1 \end{array}\right]$$

Exercise

Write the augmented matrix for:

$$\begin{aligned} y - x + z &= 7 \\ x - y - z &= 2 \\ z - y &= -1 \end{aligned}$$

Row Operations on a Matrix

The following operations on an augmented matrix will yield an equivalent matrix:

1. Interchange any two rows.
2. Multiply a row by a nonzero constant.
3. Add a multiple of one row to another row.

The above operations are denoted as follows:

1. $R_i \leftrightarrow R_j$ Interchange row i with row j
2. $cR_i \rightarrow R_i$ Multiply row i by the constant c $(\neq 0)$
3. $cR_i + R_j \rightarrow R_j$ Multiply row i by c and add to row j $(c \neq 0)$

Examples:

Perform the given operations

(a) $\left[\begin{array}{cc|c} 2 & -1 & 3 \\ 0 & 2 & 1 \end{array}\right]$ $R_1 \leftrightarrow R_2$

Soln.:

$$\left[\begin{array}{cc|c} 2 & -1 & 3 \\ 0 & 2 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \boxed{\left[\begin{array}{cc|c} 0 & 2 & 1 \\ 2 & -1 & 3 \end{array}\right]}$$

(b) $\left[\begin{array}{ccc|c} -1 & 0 & 1 & -2 \\ 3 & -1 & 2 & 3 \\ 0 & 1 & 3 & 1 \end{array}\right]$ $2R_3 \to R_3$

Soln. $2R_3 \to R_3$ means multiply third row by 2.

$$\left[\begin{array}{ccc|c} -1 & 0 & 1 & -2 \\ 3 & -1 & 2 & 3 \\ 0 & 1 & 3 & 1 \end{array}\right] \xrightarrow{2R_3 \to R_3} \left[\begin{array}{ccc|c} -1 & 0 & 1 & -2 \\ 3 & -1 & 2 & 3 \\ 0 & 2 & 6 & 2 \end{array}\right]$$

c. $\left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & 2 \\ 0 & 1 & 2 & 3 & 5 \end{array}\right]$ $R_1 - 2R_2 \to R_1$

Sln $R_1 - 2R_2 \to R_1$ means multiply row 2 by -2 and add to row 1.

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 2 & 2 \\ 0 & 1 & 2 & 3 & 5 \end{array}\right] \xrightarrow{R_1 - 2R_2 \to R_1} \left[\begin{array}{cccc|c} 1-2\cdot 0 & 2-2\cdot 1 & 0-2\cdot 2 & 2-2 3 & 2-2\cdot 2 \\ 0 & 1 & 2 & 3 & 5 \end{array}\right]$$

$$= \left[\begin{array}{cccc|c} 1 & 0 & -4 & -4 & -8 \\ 0 & 1 & 2 & 3 & 5 \end{array}\right]$$

Ques. Why does this work? In other words, why does performing the row operations yield an equivalent matrix?

Ans. Equivalent matrix means that the new matrix obtained by performing the row operations has the same solutions as the original matrix.

Let's prove it:

Say we are given an augmented matrix

$$\left[\begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{array}\right]$$

[Note: I have chosen a 2X2 matrix. The following argument works with a matrix of any order.]

This augmented matrix is equivalent to the system

$$a_{11}x + a_{12}y = b_1$$
$$a_{21}x + a_{22}y = b_2$$

Row Operation ①; Interchanging rows:

When we perform $R_1 \leftrightarrow R_2$ we get the new system

$$a_{21}x + a_{22}y = b_2$$
$$a_{11}x + a_{12}y = b_1$$

It should be clear that the new system has the same solutions as the original system. In other words, when you interchange rows, the solutions stay the same.

Row Operation ②: Multiplying a row by a constant c:
we have the system

$$\left.\begin{aligned} a_{11}x + a_{12}y &= b_1 \\ a_{12}x + a_{22}y &= b_2 \end{aligned}\right\} \text{ Ori.}$$

Say that we multiply row 2 by a nonzero-constant c, i.e. we perform $2R_2 \longrightarrow R_1$. Then we get the new system:

$$\left.\begin{aligned} a_{11}x + a_{12}y &= b_1 \\ ca_{12}x + ca_{22}y &= cb_2. \end{aligned}\right\} \text{ New}$$

Assume that (x_0, y_0) is a solution to Ori..
Then note that

$$a_{11}x_0 + a_{12}y_0$$
$$= b_1 \quad (\text{since } (x_0,y_0) \text{ is soln. to Ori.}) \quad (*)$$

and,

$$ca_{12}x_0 + ca_{22}y_0$$
$$= c(a_{12}x_0 + ca_{22}y_0)$$
$$= c\, b_2 \quad (\text{since } (x_0,y_0) \text{ is a solution to Ori.})$$

$$(**)$$

Thus, by $(*)$ and $(**)$ we see that (x_0, y_0) satisfies the new system.
Now assume that (x', y') is a solution to New, i.e. $x = x'$ and $y = y'$ satisfy New.
Then observe that

$$a_{11}x' + a_{12}y'$$
$$= b_1 \qquad (+)$$

And

$$a_{12}x' + a_{22}y'$$
$$= \frac{c}{c}(a_{12}x' + a_{22}y')$$
$$= \frac{ca_{12}x' + ca_{22}y'}{c}$$
$$= \frac{cb_2}{c} \quad (\text{since } ca_{12}x' + ca_{22}y' = cb_2)$$
$$= b_2 \qquad (++)$$

Thus, by (+) and (++) (x', y') satisfies the Ori. system. This shows that both Ori. and New have exactly the same solutions.

Row operation (3) Multiplying a row by a constant c and adding to another Row.

$$cR_i + R_j \longrightarrow R_j.$$

Proof - Bonus Exercise.

Thus, performing row operations does not change the solutions of the system. $\square$

Notice that this is exactly what we did in the elimination method.

Gauss Jordan Elimination

This is a method for solving a system of linear equations. Note that if you are given a large system, say 10 x 10 matrix, it would be cumbersome to use substitution method.

Since ________ performing row operations does not change the solutions of the system, the idea behind Gauss-Jordan Elimination is to perform the row operations and write it in a special form, namely Reduced Row Echelon Form. Note that this is essentially a generalized version of Elimination method.

Row-Echelon Form

For this form the matrix should satisfy the following:

1. Any rows consisting entirely of 0s are at the bottom of the matrix.
2. For each row that does not consist entirely of 0s, the first (leftmost) nonzero entry is 1 (called the leading 1)
3. For two successive nonzero rows, the leading 1 in the higher row is farther to the left than the leading 1 in the lower row.

Examples:

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 3 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 3 & 2 & 3 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 2 \\ 0 & 1 & -1 & 5 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 7 \end{array}\right]$$

$$\begin{bmatrix} 1 & 3 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Examples of matrices not in row-reduced row echelon form.

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 3 & 1 \end{array}\right]$$ because of the 3.

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & -3 \end{array}\right]$$ because the leading 1 on the second row is right of the leading 1 on the third row.

Advice: Just keep track of the leading 1's. Make sure everything below it is 0's. And the leading 1's have to move to the right as you go down.

Gauss-Jordan

Idea: Use row operations to transform the augmented matrix in ______- row-echelon form.

Why?

Once it is in reduced row-echelon form, we can easily find the solutions. For example, take

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

This is in reduced row-echelon form. This is equivalent to the system

$$\begin{aligned} x + 2y + 3z &= 2 \\ y + z &= 3 \\ z &= 1 \end{aligned}$$

Thus, by row 3 $z = 1$.

By row 2 $y + z = 3 \Rightarrow y = 2$

By row 1 $x + 2y + 3z = 2$

$\Rightarrow x + 4 + 3 = 2$

$\Rightarrow x = -5$

You see how easily you can find the solutions once it is in reduced row echelon form?

Take

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 3 & 2 & 1 \\ 2 & 1 & 1 & 2 \end{array}\right]$$

This is not in reduced row-echelon form. It is equivalent to

$$\begin{aligned} x + 2y + z &= 1 \\ 3y + 2z &= 1 \\ 2x + y + z &= 2 \end{aligned}$$

Not trivial as before.

Now you see why we are trying to transform to reduced row echelon form?

Ex. Apply Gauss Jordan to solve

$$\begin{aligned} x - y + 2z &= -1 \\ 3x + 2y - 6z &= 1 \\ 2x + 3y + 4z &= 8 \end{aligned}$$

Soln. The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & -1 \\ 3 & 2 & -6 & 1 \\ 2 & 3 & 4 & 8 \end{array}\right]$$

Want: Transform by row operations to reduced row echelon form.

// already have a 1 $\rightarrow$ // Will write comments enclosed in //.

need zeros here

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & -1 \\ 3 & 2 & -6 & 1 \\ 2 & 3 & 4 & 8 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & -1 \\ 3 & 2 & -6 & 1 \\ 2 & 3 & 4 & 8 \end{array}\right] \xrightarrow{R_2 - 3R_1 \to R_2} \left[\begin{array}{ccc|c} 1 & -1 & 2 & -1 \\ 0 & 5 & -12 & 4 \\ 2 & 3 & 4 & 8 \end{array}\right]$$

$$\xrightarrow{R_3 - 2R_1 \to R_3} \left[\begin{array}{ccc|c} 1 & -1 & 2 & -1 \\ 0 & 5 & -12 & 4 \\ 0 & 5 & 0 & 10 \end{array}\right]$$

(This step is not absolutely necessary but greatly reduces calculation)

$$\xleftrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & -1 & 2 & -1 \\ 0 & 5 & 0 & 10 \\ 0 & 5 & -12 & 4 \end{array}\right]$$

// $\left[\begin{array}{ccc|c} 1 & -1 & 2 & -1 \\ 0 & 5 & 0 & 10 \\ 0 & 5 & -12 & 4 \end{array}\right]$ //

need 1 here

need 0 here

$$\xrightarrow{\frac{1}{5}R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & -1 & 2 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 5 & -12 & 4 \end{array}\right]$$

$$\xrightarrow{R_3 - 5R_2 \to R_3} \left[\begin{array}{ccc|c} 1 & -1 & 2 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -12 & -6 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{12}R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & -1 & 2 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right]$$

Thus we have the system:

$$\begin{aligned} x - y + 2z &= -1 \\ y &= 2 \\ z &= \tfrac{1}{2} \end{aligned}$$

Substituting $y = 2$, $z = \frac{1}{2}$ into row 1:

$$x - 2 + 2 \cdot \frac{1}{2} = -1$$

$$\Rightarrow x - 2 + 1 = -1$$

$$\Rightarrow x = 0$$

$\therefore$ $\boxed{x = 0}$, $\boxed{y = 2}$, $\boxed{z = \frac{1}{2}}$

Ex Apply Gauss Jordan to solve:

$$2x + y = -8$$
$$x + 3y = 6$$

Soln The augmented matrix is

$$\left[\begin{array}{cc|c} 2 & 1 & -8 \\ 1 & 3 & 6 \end{array}\right]$$

$$\left/\!\!\left/ \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \right/\!\!\right/$$

want 1

want 0

want 1

$$\left[\begin{array}{cc|c} 2 & 1 & -8 \\ 1 & 3 & 6 \end{array}\right] \quad R_1 \leftrightarrow R_2 \quad \left[\begin{array}{cc|c} 1 & 3 & 6 \\ 2 & 1 & -8 \end{array}\right]$$

$$\left/\!\!\left/ \left[\begin{array}{cc|c} 1 & 3 & 6 \\ 2 & 1 & -8 \end{array}\right] \right/\!\!\right/ \quad R_2 - 2R_1 \to R_2 \quad \left[\begin{array}{cc|c} 1 & 3 & 6 \\ 0 & -5 & -20 \end{array}\right]$$

need 0

$$-\frac{1}{5} R_2 \to R_2 \quad \left[\begin{array}{cc|c} 1 & 3 & 6 \\ 0 & 1 & 4 \end{array}\right]$$

Thus we have $x + 3y = 6$
$$y = 4$$

Substituting $y = 4$ into Row 1:

$$x + 3 \cdot 4 = 6$$
$$\Rightarrow x = -6$$

$\therefore$ $\boxed{x = -6}$ $\boxed{y = 4}$

Ex: Solve using Gauss Jordan:

$$2x + y + 8z = -1$$
$$x - y + z = -2$$
$$3x - 2y - 2z = 2$$

Soln. The augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & 1 & 8 & -1 \\ 1 & -1 & 1 & -2 \\ 3 & -2 & -2 & 2 \end{array}\right]$$

$R_1 \leftrightarrow R_2$
$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 2 & 1 & 8 & -1 \\ 3 & -2 & -2 & 2 \end{array}\right]$$

$R_2 - 2R_1 \rightarrow R_2$
$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 3 & 6 & 3 \\ 3 & -2 & -2 & 2 \end{array}\right]$$

$R_3 - 3R_1 \rightarrow R_3$
$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 3 & 6 & 3 \\ 0 & 1 & -5 & 8 \end{array}\right]$$

$\frac{1}{3}R_2 \rightarrow R_2$
$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & -5 & 8 \end{array}\right]$$

$$R_3 - R_2 \to R_3 \quad \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -7 & 7 \end{array}\right]$$

$$-\frac{1}{7}R_3 \to R_3 \quad \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$\therefore$

$$\begin{aligned} x - y + z &= -2 \\ y + 2z &= 1 \\ z &= -1 \end{aligned}$$

Plugging $z = -1$ into row 2:

$$y - 2 = 1$$

$$\Rightarrow y = 3$$

Plugging $y = 3$, $z = -1$ into row 1

$$x - 3 + (-1) = -2$$

$$\Rightarrow x = 2$$

$\therefore \boxed{x = 2}, \boxed{y = 3}, \boxed{z = -1}$

Exercise Solve

$$\begin{aligned} x + y - z &= 0 \\ 2x + y + z &= 1 \\ 2x - y + 3z &= -1 \end{aligned}$$

Solve

$$
\begin{aligned}
-2x - y + 2z &= 3 \\
3x \qquad - 4z &= 2 \\
2x + y \qquad &= -1 \\
-x + y - z &= -8
\end{aligned}
$$

Solve.

$$
\begin{aligned}
x - 2y + 3z &= 1 \\
-2x + 7y - 9z &= 4 \\
x \qquad + z &= 9
\end{aligned}
$$

$$x+y-z=0$$
$$2x+y+z=1$$
$$2x-y+3z=-1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 2 & 1 & 1 & 1 \\ 2 & -1 & 3 & -1 \end{array}\right]$$

$-2R_1+R_2 \to R_2$ $$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -1 & 3 & 1 \\ 2 & -1 & 3 & -1 \end{array}\right]$$

$-2R_1+R_3 \to R_3$ $$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -1 & 3 & 1 \\ 0 & -3 & 5 & -1 \end{array}\right]$$

$(-1)R_2 \to R_2$ $$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & -3 & 5 & -1 \end{array}\right]$$

$-6+5$

$-3-1$

$3R_2+R_3 \to R_3$ $$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & -1 & -4 \end{array}\right]$$

$(-1)R_3 \to 1$ $$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

$$z=4$$
$$y-3z=-1$$
$$y-12=-1$$
$$y=11$$
$$x+11-4=0$$
$$x+7=0$$
$$x=-7$$